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COMPLETE LEFT-INVARIANT AFFINE STRUCTURES ON NILPOTENT LIE GROUPS

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0. Introduction

A smooth manifold M^n is called an affine manifold if it admits a torsion free affine connection whose curvature tensor is zero, or if it admits a coordinate system whose coordinate transition homeomorphisms are affine transformations in $\mathbf{R}^n$. For example, Riemannian flat or Lorentz flat manifolds are subclasses of such manifolds.

The manifold M is said to be complete if every geodesic can be defined on all time intervals. By a well-known theorem the connected complete affine manifolds are just the quotients $\mathbf{R}^n/\Gamma$, where Γ is a subgroup of $\mathrm{Aff}(\mathbf{R}^n)$, the group of all the affine transformations of $\mathbf{R}^n$, acting freely and properly discontinuously on $\mathbf{R}^n$.

The group theoretic nature of such Γ is an open question and it is suggested that such Γ should be virtually polycyclic (i.e., contains a subgroup of finite index which is polycyclic) (see [1], [2], [16].) Milnor showed the converse, namely every virtually polycyclic group can be realized by such a Γ [16].

Recently Fried and Goldman, following an idea of Auslander, showed that such Γ, assuming it is virtually polycyclic, is virtually contained in a connected Lie subgroup G of $\mathrm{Aff}(\mathbf{R}^n)$ which acts simply transitively on $\mathbf{R}^n$ (so G is homeomorphic to $\mathbf{R}^n$) [8]. It is well known that a simply connected Lie group G acts simply transitively on $\mathbf{R}^n$ as affine transformations iff G admits a complete left-invariant affine structure.

Conversely, if G has a complete left-invariant affine structure and Γ is a discrete subgroup, then $\Gamma \setminus G$ will become an affine manifold.

Therefore, to identify affine manifolds, it is natural to ask which simply connected Lie groups G admit a complete left-invariant affine structure. It is known that those G which act simply transitively on $\mathbf{R}^n$ as affine transformations are solvable [2], [16].

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Auslander proved that if G acts simply transitively and affinely on $\mathbf{R}^n$, the unipotent radical of the algebraic hull of G again acts simply transitively and affinely on $\mathbf{R}^n$ [2]. This suggests studying the nilpotent case first.

In fact, a compact complete affine manifold with nilpotent fundamental group is known to be a compact quotient of a nilpotent Lie group with a complete left-invariant affine structure [9].

We naturally are intereseted in two questions. The first is an existence question, i.e., whether a simply connected solvable group admits a complete affine structure, and the second is to classify the structure up to affine equivalence in low dimensions.

The first question is still an open problem. Some special cases have been known ([2], [18], [19], [16]) until Boyom recently proved the existence for general nilpotent Lie groups [3].

The general classification problem of affine structures (up to affine equivalences) on a simply connected Lie group G is very complicated. It involves the cogredience classification of vector valued bilinear forms even in the 2-step nilpotent case. Thus we restrict ourselves to the low-dimensional cases.

The dimension 2 classification was known to Kuiper about 30 years ago [15]. Braverman, in his thesis [4], classified the structure for abelian 3-dimensional group G. Vesguez showed that there are finitely many structures in dimension 4 and 5, and infinitely many in dimension $\geqslant 6$ for abelian G. Fried and Goldman obtained the classification in dimension 3 for general G [8].

The classification in dimension 4 when G is nilpotent is carried out in this paper using so-called left-symmetric (l.s.) algebra formulation.

The basic idea of the classification is to use an inductive scheme as in [8]. One difficulty in the induction in dimension 4 is the following. While a nilpotent l.s. algebra resembles a nilpotent Lie algebra in some ways, it does not necessarily have a center. In terms of a simple transitive group action, the center corresponds to the central translations in Aff($\mathbf{R}^n$). Its existence for all representations was conjectured by Auslander but Fried found a counterexample in dimension 4 [7]. In fact, we show from the classification that there are just two such nilpotent l.s. algebras without center in dimension 4.

In order to make the induction argument work, we use another characteristic ideal, $\mathfrak{g}_\infty = \cdots((\mathfrak{g}\mathfrak{g})\mathfrak{g})\cdots$, which is proper and is nonzero whenever the center of $\mathfrak{g}$ is zero.

Then, following the procedure of Lie algebra extension, we classify the l.s. algebra extension up to congruence by calculating the l.s. algebra second cohomology (this can be defined on l.s. algebras, too). We proceed to produce the classification up to isomorphism (which corresponds to affine automorphism in group terms).

This article is organized as follows. In the first section, we examine the equivalent formulation of the problem of finding complete left-invariant flat connections on a simply connected Lie group, by formulating it in terms of the l.s. algebra structure on its Lie algebra. Then in the next section we study basic structures of such an algebra when its associated Lie algebra is nilpotent. In particular, we are interested in the properties of the characteristic ideal $\mathfrak{g}_{\infty}$. Then we develop an extension theory for l.s. algebras which parallels the extension theory for Lie algebras, on which our inductive argument for classification is based. The classification question will be formulated in terms of a group action on the l.s. algebra second cohomology. In the final two sections we determine the l.s. algebras without translations, and sketch the classification for the 4-dimensional nilpotent l.s. algebras deferring all the calculations and details to a subsequent paper [14].

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1. Left-invariant flat connection and left-symmetric product

Let G be a simply connected Lie group with its Lie algebra $\mathfrak{g}$. In this section, we will reformulate the problem of finding a complete left-invariant flat connection on G in terms of a left-symmetric product on $\mathfrak{g}$ with certain properties. Most of the results in this section seem to be well known (see [2], [18], [20], [11], [16], [8], [17]) and we will reestablish some of the results necessary to set up our investigation.

We want to find a left-invariant affine connection on G which is torsion free and flat. To perceive the problem algebraically, we denote the covariant derivative $\nabla_X Y$ by XY for $X, Y \in \mathfrak{g}$ and then observe that this amounts to finding an algebra structure on $\mathfrak{g}$ with the properties

$$XY - YX = [X, Y] \quad \text{(torsion zero)}, \tag{1.1}$$

$$X(YZ) - Y(XZ) - [XY]Z = 0 \quad \text{(curvature zero)}' \tag{1.2}$$

for $X, Y, Z \in \mathfrak{g}$. Note that these two conditions imply

$$X(YZ) - (XY)Z = Y(XZ) - (YX)Z. \tag{1.3}$$

A vector space with a bilinear product which satisfies the condition (1.3) is called a *left-symmetric algebra* or l.s. algebra in short ([21], [11], [17]). Note that an associative algebra trivially satisfies (1.3). Suppose a Lie algebra admits a left-symmetric product satisfying (1.1). Then we say that this l.s. structure is *compatible* with the Lie structure. Starting with any left-symmetric algebra L, define a Lie bracket by the formula (1.1), then the Jacobi identity readily follows from the first Bianchi identity $R(XY)Z + R(YZ)X + R(ZX)Y = 0$ and we get the *associated* Lie algebra structure on L. Hence we have a one-to-one correspondence between the left-invariant flat connections on G and the compatible left-symmetric algebra structures on $\mathfrak{g}$.

Now we will examine how the complete ones can be determined in this correspondence. Suppose G admits a left-invariant flat affine structure which is complete. Then G is affine equivalent to a vector space V of the same dimension with its usual flat affine structure. Since G acts on G by left translations which is affine, the induced G-action on V by the above affine diffeomorphism will be simply transitive by affine transformations. Conversely, if G acts on V simply transitively by affine transformations, then the evaluation map ev_x at any point $x \in V$ is a diffeomorphism and the pull-back connection will be clearly left-invariant.

Consequently a complete left-invariant flat affine structure on G gives rise to a representation R of G into $\mathrm{Aff}(V) = \mathrm{Gl}(V) \circ V$, the group of affine transformations on V, with the property that ev_x is a diffeomorphism for each $x \in V$.

On the Lie algebra level, this induces a representation $\mathbf{r} = dR = (h, t)$ of $\mathfrak{g}$ into $\mathfrak{aff}(V) = \mathfrak{gl}(V) + V$ with the property that $d(\mathrm{ev}_x)$ at e (= identity of G) is an isomorphism for each $x \in V$. Conversely, since $\mathrm{ev}_{g \cdot x} = \mathrm{ev}_x \cdot r_g$, where r_g is a right translation on G, the above property implies that ev_x is a local diffeomorphism. Therefore each orbit is open, which in turn implies it is closed, and hence there is only one orbit. It follows that ev_x is a diffeomorphism since this is a covering map.

To calculate $d(\mathrm{ev}_x)$ at e explicitly, consider the evaluation map of $\mathrm{Aff}(V)$ at $x \in V$. It is easy to check that its derivative maps $(M, m) \in \mathfrak{aff}(V) = \mathfrak{gl}(V) + V$ into $Mx + m \in V \simeq T_x(V)$. Thus $d(\mathrm{ev}_x)$ at e is a linear map $\mathfrak{g} \to V$ which sends $Y \to h(Y)x + t(Y)$.

Note also that $\mathbf{r} = (h, t)$ is a Lie algebra homomorphism if and only if $h : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a Lie algebra homomorphism and t is a 1-cocycle of $\mathfrak{g}$-module V determined by h, i.e., $t[x, y] = h(x)t(y) - h(y)t(x)$.

As a summary, we have the following equivalent statements.

1.1 Proposition. *Let G be a simply connected Lie group with its Lie algebra $\mathfrak{g}$. Then the followings are equivalent.*

(1) *G admits a complete left-invariant flat affine connection.*

(2) *G acts simply transitively on a vector space V by affine transformations.*

(3) *There is a representation* $R: G \to \mathrm{Aff}(V)$ *such that the evaluation map* ev_x *is a diffeomorphism for each* $x \in V$.

(4) *There is a representation* $\mathbf{r} = (h, t): \mathfrak{g} \to \mathfrak{aff}(V) = \mathfrak{gl}(V) + V$ *such that the vector space homomorphism* $Y \to h(Y)x + t(Y)$ *is an isomorphism for each* $x \in V$.

(5) *There is a representation* $h: \mathfrak{g} \to \mathfrak{gl}(V)$ *and a linear isomorphism* $t: \mathfrak{g} \to V$ *such that* (i) $t[X, Y] = h(X)t(Y) - h(Y)t(X)$, $X, Y \in \mathfrak{g}$, *and* (ii) $Y \to h(Y)x + t(Y)$ *is an isomorphism for all* $x \in V$.

At this point, we would like to see how the original connection on G is reflected in the representation space V. If G acts simply transitively on V as affine transformations, then it can be shown by a simple calculation that the pull-back connection ∇ on G is given by $t(\nabla_X Y) = h(X)t(Y)$ with the same notations as before. Hence $\nabla_X = t^{-1} \cdot h(X) \cdot t$ and condition (ii) of (5) becomes

(1.4) $$Y \to Y + \nabla_Y X \text{ is an isomorphism for all } X \in \mathfrak{g}.$$

Observe that we recover the flatness for ∇ from the fact that $h: \mathfrak{g} \to \mathfrak{gl}(V)$ is a Lie algebra homomorphism and the torsion free condition from (i) of (5). This proves the following:

1.2 Proposition. *A left-invariant flat connection* ∇ *on a simply connected Lie group* G *is complete iff* (1.4) *holds.*

If we denote left (resp. right) multiplication by λ_X (resp. ρ_X) in the left-symmetric algebra defined by $XY = \nabla_X Y$, then the condition (1.4) can be rephrased as

(1.5) $$1 + \rho_X \text{ is an isomorphism for all } X \in \mathfrak{g}.$$

We will call a left-symmetric algebra structure with the condition (1.5) *transitive* as in [17]. With this terminology, we conclude that the notion of the complete left-invariant flat connection on G coincides with that of the transitive compatible left-symmetric algebra structure on its Lie algebra $\mathfrak{g}$.

We close this section by a well-known observation that a linear Lie group which acts simply transitively as affine transformations on a vector space V is diffeomorphic to V and hence is solvable (see [2], [16], [11], [17].)

2. Nilpotent Lie algebra with left-symmetric product

We already know that a Lie algebra which admits a compatible transitive l.s. product has to be solvable. In this article, we will specialize to the case of a nilpotent Lie algebra. This section is devoted to establishing some basic structure theorems which will be used later. One important theorem was first proved by Scheuneman. He showed that a nilpotent Lie group which acts simply transitively on $\mathbf{R}^n$ by affine transformations is unipotent [20]. We restate his theorem in terms of a l.s. product and give another proof in this context. Let L be a Lie algebra. Recall that if L admits a compatible l.s. product, then left multiplication λ_x induces a Lie algebra homomorphism $\lambda: L \to \mathfrak{gl}(L)$ defined by $x \to \lambda_x$, and this product is transitive iff $1 + \rho_x$ is an isomorphism for all $x \in L$.

2.1 Theorem (*Scheuneman*). *If L is a transitive l.s. algebra whose associated Lie algebra is nilpotent, then the left multiplication λ_x is nilpotent for all $x \in L$.*

Proof. Let $L_{\mathbf{C}}$ be the complexification of L so that $L_{\mathbf{C}}$ has the induced l.s. product. Then λ has an obvious extension: $L_{\mathbf{C}} \to \mathfrak{gl}(L_{\mathbf{C}})$ which is still denoted by λ. Since $\lambda(L_{\mathbf{C}})$ is a nilpotent Lie algebra, as is well known $\lambda(L_{\mathbf{C}})$ can be decomposed as a direct sum of its weight spaces V_α as $\lambda(L_{\mathbf{C}})$-submodules, i.e. $\alpha \in L_{\mathbf{C}}^* = \mathrm{Hom}_{\mathbf{C}}(L_{\mathbf{C}}, \mathbf{C})$ and $V_\alpha = \{ y \in L_{\mathbf{C}} \mid (\lambda_x - \alpha(x))^m y = 0 \text{ for some } m\}$. By Lie's Theorem, there is a basis $\{a_1, \cdots, a_k\}$ in V_α with respect to which

$$\lambda_x \mid V_\alpha = \begin{bmatrix} \alpha(x) & & * \\ & \ddots & \\ 0 & & \alpha(x) \end{bmatrix}$$

(see, for example, [13, p. 50].) Hence if we let $V_\alpha^j = \mathrm{span}\langle a_1, \cdots, a_j \rangle$, then V_α^j becomes a left ideal with respect to l.s. product and $x \cdot a_i \equiv \alpha(x) a_i \pmod{V_\alpha^{i-1}}$.

Now we want to show $\alpha = 0$. Let $\{b_1, \cdots, b_l\}$ be such a basis for V_β as above. Then

$$[b_i, a_j] = b_i a_j - a_j b_i \in \alpha(b_i) a_j + V_\alpha^{j-1} + V_\beta^i \subset V_\alpha^j + V_\beta^i.$$

Thus

$$[b_i, V_\alpha^j] \in V_\alpha^j + V_\beta^i,$$

$$[b_i [b_i a_j]] \equiv \alpha(b_i)^2 a_j \quad \left(\mathrm{mod}\, V_\alpha^{j-1} + V_\beta^i\right),$$

$$[[b_i \cdots [b_i a_j]] \cdots] \equiv \alpha(b_i)^N a_j \quad \left(\mathrm{mod}\, V_\alpha^{j-1} + V_\beta^i\right).$$

Since $L_{\mathbf{C}}$ is nilpotent, $\alpha(b_i) = 0$ if $\alpha \neq \beta$. The same argument applied to $[a_i, a_j]$ with $i < j$ shows that $\alpha(a_i) = 0$, $i = 1, \cdots, k-1$.

It only remains to show that $\alpha(a_k) = 0$. First note that $\bar{\alpha} \in L_{\mathbf{C}}^*$ defined by $\bar{\alpha}(\bar{x}) = \overline{\alpha(x)}$ is also a weight for $\lambda(L_{\mathbf{C}})$ since $(\lambda_x - \alpha(x))^m y = 0$ implies $(\lambda_{\bar{x}} - \overline{\alpha(x)})^m \bar{y} = 0$, and $\{\bar{a}_1, \cdots, \bar{a}_k\}$ becomes the associated basis for $V_{\bar{\alpha}}$. Suppose $\alpha \neq \bar{\alpha}$, then $\alpha(\bar{a}_k) = 0$ from above. Thus we have $a_k \cdot a_k \equiv \alpha(a_k) a_k$, $\bar{a}_k \cdot a_k \equiv 0 \pmod{V_\alpha^{k-1}}$ and so $\pmod{V_\alpha^{k-1} + V_{\bar{\alpha}}^{k-1}}$. Let us denote $a_k = u + iv$, $u, v \in L_{\mathbf{R}}$, and $\alpha(a_k) = p + iq$, $p, q \in \mathbf{R}$. If $u \in V_\alpha^{k-1} + V_{\bar{\alpha}}^{k-1}$, then $\alpha(u) = 0$ and hence $\alpha(a_k) = 0$ since $\alpha(\bar{a}_k) = 0$. Hence assume $u \notin V_\alpha^{k-1} + V_{\bar{\alpha}}^{k-1}$.

Since $V_\alpha^{k-1} + V_{\bar\alpha}^{k-1}$ is invariant under complex conjugation, we get, by comparing real and imaginary parts,

$$u^2 - v^2 \equiv pu - qv, \qquad uv + vu \equiv qu + pv,$$
$$u^2 + v^2 \equiv 0, \qquad vu \equiv uv \quad \left(\operatorname{mod} V_\alpha^{k-1} + V_{\bar\alpha}^{k-1}\right).$$

It follows that $2u^2 \equiv pu - qv$, $2uv \equiv qu + pv$, and hence $u(2pu + 2qv) = 2pu^2 + 2quv \equiv (p^2 + q^2)u$. This implies $(-(p^2 + q^2) + \rho_x)(u) \equiv 0$, where $x = 2pu + 2qv \in L_{\mathbf{R}}$. Note that both ρ_{a_k} and $\rho_{\bar a_k}$ and hence ρ_x sends $V_\alpha^{k-1} + V_{\bar\alpha}^{k-1}$ into itself, and $u \notin V_\alpha^{k-1} + V_{\bar\alpha}^{k-1}$. Therefore $p^2 + q^2 = 0$ and $\alpha(a_k) = 0$, otherwise $-(p^2 + q^2) + \rho_x$ would be an isomorphism by transitivity (see (1.5)).

Finally, if $\alpha = \bar\alpha$, then α is real, i.e. $\alpha : L_{\mathbf{R}} \to \mathbf{R}$. Thus

$$u \cdot a_k \equiv \alpha(u)a_k, \quad v \cdot a_k \equiv \alpha(v)a_k \quad \left(\operatorname{mod} V_\alpha^{k-1} + V_{\bar\alpha}^{k-1}\right)$$

for $\alpha(u)$, $\alpha(v) \in \mathbf{R}$. It follows that $u \cdot u \equiv \alpha(u)u$, $v \cdot v \equiv \alpha(v)v$, and again $\alpha(u) = \alpha(v) = 0$ and $\alpha(a_k) = 0$ by transitivity. q.e.d.

It is also known that the converse of this theorem holds and, furthermore, ρ_x is nilpotent (see [11]). This of course implies L is transitive (see [10] for more about completeness). We will give another proof here.

2.2 Theorem. *If L is an l.s. algebra where the left multiplication λ_x is nilpotent for all $x \in L$, then its associated Lie algebra is nilpotent and the right multiplication ρ_x is nilpotent for all $x \in L$.*

Proof. Suppose λ_x is nilpotent for all $x \in L$. We want to show that $L, L^{(2)} = [L, L], \cdots, L^{(k+1)} = [L, L^{(k)}], \cdots$ stops at 0 at some stage. Suppose A and B are subspaces of L. Then the subspace of L which is spanned by the elements of the form $a \cdot b$, $a \in A$, $b \in B$, will be denoted by $A \cdot B$. Since λ_x is nilpotent for all $x \in L$, $\lambda(L)$ can be put into strict upper triangular matrices simultaneously by Engel's theorem. Therefore the decreasing sequence $L^1 = L, L^2 = L \cdot L, \cdots, L^{i+1} = L \cdot L^i, \cdots$ will stop at 0 at some stage. Now we can complete the proof of the first part from the following claim.

Claim. (1) $L^{(k)} \cdot L^l \subset L^{k+l} \cdot (l > 0)$.

(2) $L^{(k)} \subset L^k$.

Proof of (1). Apply induction on k.

$$\begin{aligned} L^{(k+1)} \cdot L^l &= [L, L^{(k)}] \cdot L^l \\ &\subset L \cdot (L^{(k)} \cdot L^l) - L^{(k)}(L \cdot L^l) \quad \text{(by left-symmetry)} \\ &\subset L \cdot L^{k+l} - L^{(k)} \cdot L^{l+1} \quad \text{(by induction hypothesis)} \\ &\subset L^{k+l+1}. \end{aligned}$$

Proof of (2). Again apply induction on k.

$$L^{(k+1)} = [L, L^{(k)}] \subset L \cdot L^{(k)} - L^{(k)} \cdot L$$
$$\subset L \cdot L^{k} - L^{(k)} \cdot L \quad \text{(by induction hypothesis)}.$$

Now note that $L^{(k)} \cdot L \subset L^{k+1}$ by (1).

For the second part, first note that the left-symmetry of the product is equivalent to the condition

$$(2.1) \qquad [\lambda_x, \rho_y] = \rho_{xy} - \rho_y \rho_x, \qquad x, y \in L.$$

It follows from this that $\rho_x^2 = \rho_{x^2} - [\lambda_x, \rho_x]$ and the induction argument shows

$$\rho_x^n = \rho_{x^n} - [\lambda_{x^{n-1}}, \rho_x] - \rho_x[\lambda_{x^{n-2}}, \rho_x] - \cdots - \rho_x^{n-2}[\lambda_x, \rho_x].$$

Take the trace of both sides and note that $\operatorname{tr}[A, B] = 0$ and $\operatorname{tr}(A[BC]) = \operatorname{tr}([AB]C)$. We obtain $\operatorname{tr}(\rho_x^n) = \operatorname{tr}(\rho_{x^n})$. Observe that $\operatorname{tr}(\rho_x) = 0$ for all $x \in L$ since $\operatorname{ad}_x = \lambda_x - \rho_x$. Consequently $\operatorname{tr}(\rho_x^n) = 0$ for all n and ρ_x is nilpotent. q.e.d.

By virtue of this theorem, we propose the following definition.

Definition. Let L be an l.s. algebra. Then L is called *nilpotent* if λ_x is nilpotent for all $x \in L$.

Remark. This definition is different from that used in [11] and [17], where an l.s. algebra is called nilpotent when simply ρ_x is nilpotent for all $x \in L$.

From now on we will assume that L is a nilpotent transitive l.s. algebra. We saw in the proof of Theorem 2.2 that the decreasing sequence $L^1 = L, L^2 = L \cdot L, \cdots, L^{i+1} = L \cdot L^i, \cdots$ stops at 0 at some stage. But if we consider successive right multiplications, then the situation is not the same because $\rho : L \to \mathfrak{gl}(L)$ is not a Lie algebra homomorphism in general. Let us consider the decreasing sequence

$$L_2 = L \cdot L,\ L_3 = L_2 \cdot L, \cdots, L_{i+1} = L_i \cdot L, \cdots.$$

If $L_k = L_{k+1}$ for some k, then this sequence will stabilize from that stage on and we will denote this limit by L_∞. It is easy to show using the left-symmetry that all L_i's are two-sided ideals with respect to the product structure. The ideal L_∞ will play an important role later in this paper.

2.3 Proposition. (1) *There is a basis* $\{e_1, \cdots, e_r, e_{r+1}, \cdots, e_n\}$ *for* L *such that* $L_\infty = \operatorname{span}\langle e_1, \cdots, e_r\rangle$, *and the matrices of* $\rho(L)$ *are of the block form*

$$\begin{bmatrix} A_r & * \\ 0 & B_{n-r} \end{bmatrix}$$

and the matrices of $\lambda(L)$ *are of the form*

$$\begin{bmatrix} C_r & * \\ 0 & D_{n-r} \end{bmatrix}$$

simultaneously, where B_{n-r}, C_r, D_{n-r} *are strict upper triangular matrices.*

(2) *If* $L_\infty = 0$, *then* $\rho(L)$ *and* $\lambda(L)$ *can be represented as strict upper triangular matrices simultaneously.*

Proof. Consider the sequence $L_1, L_2, \cdots$, defined above. Since these are two-sided ideals, they are invariant under all ρ_x and λ_x, and hence get induced maps $\bar{\rho}_x$ and $\bar{\lambda}_x$ of L_i/L_{i+1} into itself. From the definition of L_i, $\bar{\rho}_x = 0$ and Engel's theorem gives simultaneous triangular forms for $\bar{\lambda}_x$, whence (1) follows. (2) is immediate from (1).

Now we introduce another important (two-sided) ideal

$$T = \ker \lambda = \{ a \in L \mid a \cdot x = 0 \text{ for all } x \in L \}.$$

This is clearly a Lie ideal since λ is a Lie homomorphism, and is trivially a right ideal. If $L_\infty = 0$, then from 2.3 above, $0 = \rho_x(e_1) = e_1 \cdot x$ for all $x \in L$. Hence $e_1 \in T$ and we have the following

2.4 Corollary. *If* $L_\infty = 0$, *then* $T \neq 0$.

We collect some more properties of L_∞ and T which will be useful later.

2.5 Proposition. (1) L_∞ *is a proper ideal.*

(2) $\Sigma_x \rho_x(L_\infty) = L_\infty$.

(3) $\dim L_\infty \neq 1$.

Proof. $L \cdot L$ is a proper ideal of L since $\lambda(L)$ is strict upper triangular. (1) follows from this and $L_\infty \subset L \cdot L$. (2) is just $L_\infty \cdot L = L_\infty$. If $\dim L_\infty = 1$, then $\rho_x \mid L_\infty$ being nilpotent is a zero map for all $x \in L$, which contradicts (2).

2.6 Lemma. *If R is a 2-dimensional (two-sided) ideal of L, then $R \cdot L$ is a proper subspace of R.*

Proof. By abusing the notation, $\lambda : L \to \mathfrak{gl}(R)$ is a Lie algebra homomorphism, and so $\dim \lambda(L) \leqslant 1$ by Engel's theorem. It follows that $L = K + V_1$ (vector space direct sum), where $\lambda \mid V_1$ is an isomorphism and hence $V_1 \simeq \mathbf{R}$ or 0 and $K = \ker \lambda = \{a \in L \mid a \cdot x = 0 \text{ for all } x \in R\}$. Consider the restriction of ρ onto K, $\rho : K \to \mathfrak{gl}(R)$. Then

$$\rho_{xy}(r) = r(xy) = x(ry) + (rx)y - (xr)y = (rx)y = \rho_y \rho_x(r)$$

for $x, y \in K$, $r \in R$ (use left-symmetry for the second equality). This shows that $\rho_{[xy]} = -[\rho_x, \rho_y]$, i.e., $-\rho$ is a Lie algebra homomorphism. Thus $\dim \rho(K) \leqslant 1$, again by Engel's theorem. As before $K = J + V_2$, where $J = \ker \rho$ and $V_2 \simeq \mathbf{R}$ or 0. Finally, $L = J + V_2 + V_1$. Denote $V_i = \operatorname{span}\langle v_i \rangle$. Suppose $R \cdot L$ is not proper; $R = R \cdot L = R \cdot (V_2 + V_1) = R \cdot v_1 + R \cdot v_2$. Then each $v_i \neq 0$

since multiplication is nilpotent, and $R \cdot v_i = \text{span}\langle r_i \rangle$ for some $\{r_1, r_2\}$ which is linearly independent. Note that $r_1 \cdot v_1 = 0$ and $r_2 \cdot v_2 = 0$ since ρ_{v_i} is nilpotent, and this implies $r_2 \cdot v_1 = \beta r_1$ and $r_1 \cdot v_2 = \alpha r_2$ for some nonzero α, β. Therefore $\rho_{v_1+v_2}$ sends $r_1 \to \alpha r_2$ and $r_2 \to \beta r_1$ and defines a nonsingular map contradicting its nilpotency.

2.7 Corollary. (1) $\dim L_\infty \neq 2$.

(2) *If* $\dim L = 3$, *then* $T \neq 0$.

If we denote the center of Lie algebra L by $Z = Z(L)$, then $T \neq 0$ implies $C = T \cap Z \neq 0$ by the well-known fact for nilpotent Lie algebra (see, for example, [12, p. 13]). If L is a Lie algebra $\mathfrak{g}$ of a nilpotent Lie group G, then in terms of simple transitive action, $\mathfrak{g} \to \mathfrak{aff}(\mathfrak{g})$ defined by $X \to (\lambda_X, X)$ induces $G \to \text{Aff}(\mathfrak{g})$ and the exponential of C corresponds exactly to central translations. Consequently, in group term, 2.7(2) simply says that if G is a 3-dimensional nilpotent Lie group which acts simply transitively on $\mathbf{R}^3$, then it contains a nontrivial central translation. This fact is known to Fried-Goldman. (Compare [18, Proposition 3.5].)

3. Extensions of left-symmetric algebras

In this section we will consider the extension problem of l.s. algebras by imitating the well-known procedure for groups or Lie algebras. The case of l.s. algebras is more complicated and it is not clear to us that a full cohomology theory can be developed as in the Lie algebra case. We define only the 2-dimensional cohomology and establish the 1-1 correspondence with the congruence classes.

We concentrate mainly on the central extension case because this is useful in determining the nilpotent l.s. algebras inductively. In order to classify the l.s. algebras up to isomorphism, we consider the actions of automorphism groups on the central extensions so that the orbit space of this action corresponds to the isomorphism classification (see [14] for more details and information).

Suppose we have two l.s. algebras A and K; we want to define an l.s. structure on a vector space extension L of A by K so that

$$0 \to A \xrightarrow{i} L \xrightarrow{p} K \to 0$$

becomes a short exact sequence in the category of l.s. algebras.

Choose a linear map $u: K \to L$ with $p \circ u = \text{id}$ and denote $u(x) = \bar{x}$. Then an element of L can be written uniquely as $a + \bar{x}$, $a \in A$, $x \in K$, and

$$(a + \bar{x}) \cdot (b + \bar{y}) = a \cdot b + \bar{x} \cdot b + a \cdot \bar{y} + \bar{x} \cdot \bar{y}. \tag{3.1}$$

This product is left-symmetric, i.e. $R(rs)t = r(st) - s(rt) - [r, s]t = 0$ with $r = a + \bar{x}$, $s = b + \bar{y}$, $t = c + \bar{z}$ iff

$$\text{(3.2)} \qquad \begin{array}{lll} \text{(i) } R(ab)c = 0, & \text{(ii) } R(\bar{x}b)c = 0, & \text{(iii) } R(ab)\bar{z} = 0, \\ \text{(iv) } R(\bar{x}\bar{y})c = 0, & \text{(v) } R(a\bar{y})\bar{z} = 0, & \text{(vi) } R(\bar{x}\bar{y})\bar{z} = 0. \end{array}$$

Note that we do not have to worry about the two missing conditions $R(a\bar{y})c = 0$ and $R(\bar{x}b)\bar{z} = 0$ because of the identity $R(rs)t = -R(sr)t$. Also observe that (i) is just the left-symmetry of A.

Since $p(\bar{x} \cdot \bar{y}) = x \cdot y = p(\overline{x \cdot y})$, $\bar{x} \cdot \bar{y} = \overline{x \cdot y} + g(x, y)$ for a bilinear map $g: K \times K \to A$. Define a linear map $\lambda, \rho: K \to \mathfrak{gl}(A)$ by $\lambda_x(b) = \bar{x} \cdot b$ and $\rho_y(a) = a \cdot \bar{y}$. Then the conditions (3.2)(ii)–(v) become

(3.3)

$$\begin{array}{ll} \text{(ii)} & \Leftrightarrow \lambda_x(bc) = \lambda_x(b) \cdot c + b \cdot \lambda_x(c) - \rho_x(b) \cdot c, \\ \text{(iii)} & \Leftrightarrow \rho_z[ab] = a \cdot \rho_z(b) - b \cdot \rho_z(a), \\ \text{(iv)} & \Leftrightarrow [\lambda_x \lambda_y] - \lambda_{[xy]} = \lambda_{f(x,y)}, \quad \text{where } f(x, y) = g(x, y) - g(y, x), \\ \text{(v)} & \Leftrightarrow [\lambda_y \rho_z] + \rho_z \rho_y - \rho_{yz} = \rho_{g(y,z)}. \end{array}$$

Condition (vi) can be written as $\delta g = 0$, where

$$\text{(3.4)} \qquad \begin{aligned} \delta g(x, y, z) = {} & g(x, y \cdot z) - g(y, x \cdot z) - g([xy], z) \\ & + \bar{x} \cdot g(y, z) - \bar{y} \cdot g(x, z) - f(x, y) \cdot \bar{z}, \end{aligned}$$

where $f(x, y)$ is as in (3.3)(iv).

3.1 Proposition. *There exists an l.s. algebra structure on a vector space L extending an l.s. algebra A by an l.s. algebra K iff there are linear maps $\lambda, \rho: K \to \mathfrak{gl}(A)$ and a bilinear map $g: K \times K \to A$ such that $\delta g = 0$ and* (3.3) *holds.*

If $A = E$ is a trivial l.s. algebra, i.e. all the products are 0, then (3.3) simplifies to

$$\text{(3.5)} \qquad \begin{array}{l} (1)\ \lambda \text{ is a Lie algebra homomorphism (from (3.3)(iv))}, \\ (2)\ [\lambda_y, \rho_z] + \rho_z \rho_y - \rho_{yz} = 0 \quad \text{(from (3.3)(v))} \end{array}$$

and this gives a "K-bimodule" structure on E.

Let E be a K-bimodule defined by (3.5). Then define a coboundary operator $\delta_1: \mathscr{L}(K, E) \to \mathscr{L}^2(K, E)$ and $\delta_2: \mathscr{L}^2(K, E) \to \mathscr{L}^3(K, E)$ by

$$\text{(3.6)} \qquad \begin{array}{l} h \in \mathscr{L}(K, E) \Rightarrow \delta_1 h(x, y) = hx \cdot y + x \cdot hy - h(x \cdot y), \\ g \in \mathscr{L}^2(K, E) \Rightarrow \delta_2 g(x, y, z) = \delta g(x, y, z) \quad \text{defined in (3.4)}, \end{array}$$

where $\mathscr{L}^n(K, E)$ is the space of E-valued n-linear forms. Direct calculation shows that $\delta^2 = 0$ and this defines cohomology in dimension 2, $H^2_{\lambda,\rho}(K, E) = Z^2_{\lambda,\rho}/B^2_{\lambda,\rho}$.

It can be shown that the congruence classes of the extensions of E by K is in 1-1 correspondence with $H^2_{\lambda,\rho}(K, E)$ as in the case of group or Lie algebra extension theory.

We call $C = C(L) = \{r \in L \mid r \cdot s = 0 = s \cdot r, \text{ for all } s \in L\} = Z(L) \cap T(L)$ the *center* of l.s. algebra L. (Recall that $T(L)$ is the *left kernel* of L defined by $T(L) = \{r \in L \mid r \cdot s = 0 \text{ for all } s \in L\}$ and $Z(L)$ is the center of Lie algebra L.) An l.s. extension $0 \to V \to L \to K \to 0$ is called *central* if $i: V \to C(L)$, and in this case the K-bimodule structure on V is trivial, i.e., $\lambda = \rho = 0: K \to \mathfrak{gl}(V)$. Since actions λ and ρ are trivial, the coboundary operation δ has a simpler formula:

$$\begin{aligned} &\delta h(x, y) = -h(x \cdot y), \qquad h \in \mathscr{L}(K, V), \\ (3.7) \qquad &\delta g(x, y, z) = g(x, y \cdot z) - g(y, x \cdot z) - g([x, y], z), \\ &\qquad\qquad g \in \mathscr{L}^2(K, V). \end{aligned}$$

3.2 Proposition. *Given K an l.s. algebra and V a trivial K-bimodule, a central l.s. extension L exists iff there exists a bilinear map $g: K \times K \to V$ such that $\delta g = 0$. In this case the product on L is defined by $(a, x) \cdot (b, y) = (g(x, y), x \cdot y)$. Furthermore, congruence classes of the central extensions are in 1-1 correspondence with*

$$H^2(K, V) = \{g \in \mathscr{L}^2(K, V) \mid \delta g = 0\}/\{\delta h \mid h \in \mathscr{L}(K, V)\}.$$

When a l.s. algebra K and a trivial K-bimodule V are understood, we denote a central extension corresponding to a class $[g] \in H^2(K, V)$ by $0 \to V \to L \to K \to 0$; $[g] \in H^2(K, V)$, or simply by $L_{[g]}$. Let $I_g = N(g) \cap C(K)$, where $N(g) = \{x \in K \mid g(x, y) = 0 = g(y, x) \text{ for all } y \in K\}$, the kernel of the bilinear form g. Then it is easy to show that $(a, x) \in C(L)$ iff $x \in I_g$, and hence $V = C(L)$ iff $I_g = 0$. Moreover, $I_g = I_{g+\delta h}$ and $I_{[g]}$ is well defined depending only on the cohomology class of g.

Let $L = L_{[g]}$ and $L' = L_{[g']}$ be two central extensions of V by K. If $\alpha \in \text{Aut}(V) = \text{Gl}(V)$ and $\eta \in \text{Aut}(K)$, then (α, η) defines an isomorphism $\theta: (a, x) \to (\alpha(a) + \beta(x), \eta(x))$ from L onto L' provided $\eta^* g' = \alpha_* g - \delta\beta$, i.e., $\eta^*[g'] = \alpha_*[g]$. This suggests to us to define an action of $G = \text{Aut}(V) \times \text{Aut}(K)$ on $H^2(K, V)$ by $(\alpha, \eta) \cdot [g] = \alpha_*(\eta^{-1})^*[g]$. Denoting the orbit of $[g]$ by $G[g]$, we can obtain

3.3 Proposition. (1) $G[g] = G[g'] \Rightarrow L_{[g]} \cong L_{[g']}$

(2) $G[g] = G[g'] \Leftrightarrow L_{[g]} \cong L_{[g']}$ *holds on $E(K, V)$, where $E(K, V) = \{[g] \in H^2(K, V) \mid I_{[g]} = 0\}$ = central extensions with $C(L) = V$.*

This proposition says that the classification of the exact central extensions of V by K up to l.s. isomorphism is simply the orbit space of $E(K, V)$ under the obvious action of $G = \mathrm{Aut}(V) \times \mathrm{Aut}(K)$.

4. 4-dimensional nilpotent l.s. algebra without translations

Let L be a nilpotent l.s. algebra. If $\dim L = 2$ or 3, then we know $L_\infty = 0$ from 2.5 and 2.7, and hence $T(L) \neq 0$ by 2.4, which implies $C(L) = T(L) \cap Z(L) \neq 0$. Now we have an l.s. short exact sequence, $0 \to C \to L \to K \to 0$ with $\dim K \leqslant 2$. First determine the 2-dimensional L to show $L = \mathbf{R}^2$ or $N^2 = \langle e_1, e_2 \mid e_2 \cdot e_2 = e_1 \rangle$, and then the 3-dimensional L inductively: Since $\dim K = 1$ or 2, we can easily calculate $H^2(K, C)$ and $G = \mathrm{Aut}(C) \times \mathrm{Aut}(K)$ from which we can obtain the classification as given in 3.3.

Unlike the 2- or 3-dimensional case, the 4-dimensional L does not necessarily have a nontrivial $T(L)$, hence a nontrivial $C(L)$. In fact, it was conjectured by L. Auslander [2] that $T(L) \neq 0$ for all dimensions and D. Fried discovered a counterexample in the dimension 4 [7]. (Note that in terms of simple transitive actions, $T(L)$ corresponds to the translations. See the comments following Corollary 2.7.) This type of l.s. algebra adds one more difficulty to the 4-dimensional classification problem and will be described completely in this section.

First of all recall that if $T(L) = 0$, then $L_\infty \neq 0$ (Corollary 2.4). Moreover, $\dim L_\infty = 3$ from 2.5 and 2.7. We will denote L_∞ by R for simplicity. Thus we have a short exact sequence of l.s. algebras,

$$0 \to R \to L \to \mathbf{R} \to 0, \qquad R = L_\infty. \tag{4.1}$$

We choose a section $u : \mathbf{R} \to L$ as before and let $u(1) = x$. Define $\lambda, \rho \in \mathfrak{gl}(R)$ by $\lambda(r) = x \cdot r$ and $\rho(r) = r \cdot x$, and let $x \cdot x = s \in R$. Hence λ and ρ are nilpotent linear transformations of R such that

$$(4.2) \qquad \begin{aligned} &(1)\ \lambda(a \cdot b) = \lambda(a) \cdot b + a \cdot \lambda(b) - \rho(a) \cdot b, \\ &(2)\ \rho[ab] = a \cdot \rho(b) - b \cdot \rho(a), \\ &(3)\ [\lambda, \rho] + \rho^2 = \rho_s. \end{aligned}$$

(see Proposition 3.1). Moreover the necessary and sufficient conditions for $T(L) = 0$ in terms of λ, ρ and s can be given as follows.

(4) $\rho(t) \neq 0 \quad$ for $t \in T(R)$,

(5) There is no $r \in R$ with the properties $\lambda_r = \lambda$ and $\rho(r) = s$.

In fact, an element $p \in L$ can be written as $r + \alpha x$ with $r \in R$. Note that $r \in T(L)$ iff $r \in T(R)$ and $\rho(a) = 0$, and $-r + x \in T(L)$ iff $\lambda_r = \lambda$ and $\rho(r) = s$.

4.1 Lemma. *R, as a Lie algebra, is the Heisenberg algebra.*

Proof. Suppose R is an abelian Lie algebra. Then $a \cdot \rho(b) = b \cdot \rho(a)$ from (2). Given $t \in T(R)$, $\rho(t) \cdot r = r \cdot \rho(t) = t \cdot \rho(r) = 0$ for all $r \in R$, and so $\rho(t) \in T(R)$. From (4), $\rho(t) \neq 0$ and we get a contradiction to the nilpotency of ρ.

4.2 Lemma. *There is a basis* $\mathbf{e} = \{e_1, e_2, e_3\}$ *in R such that the structure of R is given by* $\langle e_1, e_2, e_3 \mid e_2 \cdot e_3 = e_1, e_3 \cdot e_3 = e_2\rangle$ *and*

$$\lambda = \begin{bmatrix} 0 & 0 & a \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad a \in \mathbf{R},$$

$$\rho = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \quad \textit{with respect to } \mathbf{e}.$$

Proof. Since R is Heisenberg, $\dim Z(R) = 1$ and $0 \neq C(R) \subset Z(R)$ implies that $C(R) = Z(R) = \langle c \rangle$. From (4), $\rho(c) \neq 0$ and hence $\{c, \rho(c)\}$ is linearly independent since ρ is nilpotent. Since $\phi = \lambda - \rho$: $Z(R) \to Z(R)$ is nilpotent, $\phi(c) = 0$ and $\lambda(c) = \rho(c)$. Now let $K = \{k \in R \mid r \cdot k = 0$ for all $r \in R\}$ Then we claim that $K = \langle c, \rho(c) \rangle$. Clearly $c \in K$ and $0 = \lambda(r \cdot c) = \phi(r) \cdot c + r \cdot \lambda(c)$ implies $\lambda(c) = \rho(c) \in K$. Note that $K \neq R$, otherwise R would be a trivial l.s. algebra and hence an abelian Lie algebra. Thus $\dim K = 2$ and the claim follows.

$$[\lambda, \rho] + \rho^2 = \rho_s \Rightarrow \lambda(\rho(c)) - \rho(\lambda(c)) + \rho(\rho(c)) = c \cdot s = 0$$
$$\Rightarrow 0 = \lambda(\rho(c)) = \lambda^2(c).$$

This shows $\lambda : K \to K$ and $\lambda = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$ with respect to $\{c, \rho(c)\}$. On the other hand, we claim $\rho^2(c) \notin K$. In fact, if we assume $\rho^2(c) \in K$, then $\rho : K \to K$ and $\rho = \left[\begin{smallmatrix} 0 & * \\ 1 & * \end{smallmatrix}\right]$ relative to $\{c, \rho(c)\}$. Since ρ is nilpotent, $\rho = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$ and $\rho^2(c) = 0$. Again the nilpotency of ρ implies $\rho : R \to K$ and so $\rho[r, r'] = r \cdot \rho(r') - r' \cdot \rho(r) = 0$ for $r, r' \in R$. As R is Heisenberg, $c = [r, r']$ for some $r, r' \in R$ and $\rho(c) = 0$, which contradicts (4), and whence the claim follows. Therefore $\{c, \rho(c), \rho^2(c)\}$ form a basis for R and let $\mathbf{e} = \{e_1 = -c,\ e_2 = -\rho(c), e_3 = \rho^2(c)\}$. It is clear the nilpotency of λ and ρ implies that

$$\rho = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \quad \text{and} \quad \lambda = \begin{bmatrix} 0 & 0 & a \\ 1 & 0 & b \\ 0 & 0 & 0 \end{bmatrix}$$

with respect to $\mathbf{e}$.

Since $C(R) = \langle e_1 \rangle$ and $K(R) = \langle e_1, e_2 \rangle$, the only possible nonzero products will be $e_2 \cdot e_3$ and $e_3 \cdot e_3$. Observe that $\phi(e_2) = \lambda(e_2) - \rho(e_2) = e_3$ and $\lambda(e_3) \in K$ to obtain

$$(*) \qquad \lambda(e_2 \cdot e_3) = \phi(e_2) \cdot e_3 + e_2 \cdot \lambda(e_3) = e_3 \cdot e_3.$$

If $e_2 \cdot e_3 = 0$, then $e_3 \cdot e_3 = 0$ and R would be a trivial algebra. Hence $e_2 \cdot e_3 \neq 0$ and this implies with $e_3 \cdot e_2 = 0$ that $[e_2, e_3] \neq 0$ which is an element of $[R, R] = Z(R) = \langle e_1 \rangle$ and $e_2 \cdot e_3 = \alpha e_1$ for $\alpha \neq 0$. We may assume $\alpha = 1$ by defining new basis "e_i" to be $(1/\alpha)e_i$. Now from $(*)$, $e_2 = \rho(e_1) = \lambda(e_1) = \lambda(e_2 \cdot e_3) = e_3 \cdot e_3$, and $R = \langle e_1, e_2, e_3 \mid e_2 \cdot e_3 = e_1, e_3 \cdot e_3 = e_2 \rangle$. Similarly, consider $\lambda(e_3 \cdot e_3) = \phi(e_3) \cdot e_3 + e_3 \cdot \lambda(e_3) = (ae_1 + be_2) \cdot e_3 = be_1$ and $\lambda(e_3 \cdot e_3) = \lambda(e_2) = 0$. Thus $b = 0$ in the matrix representation of λ and the lemma is proved. q.e.d.

Suppose we have a 4-dimensional nilpotent l.s. algebra L. Then we get an extension of the form (4.1). The structure of $R, \lambda = \lambda_x$, and $\rho = \rho_x$ are determined as in 4.2, and these are the necessary conditions. Conversely, suppose we have a 3-dimensional nilpotent l.s. algebra R of the structure given in 4.2. Then it is easy to check that λ and ρ given in 4.2 satisfy (4.2)(1), (2) and also (4), (5). If we define $s = pe_1 + qe_2 - ae_3$ for any p, q, then we can immediately show that the condition (3) is also satisfied. Hence we get a l.s. structure on L (by Proposition 3.1) with $T(L) = 0$. Now let us write down the structure of L explicitly using a basis. From 4.2, we have

$$e_2 \cdot e_3 = e_1, \quad e_3 \cdot e_3 = e_2; \qquad x \cdot e_1 = e_2, \quad x \cdot e_3 = ae_1;$$
$$e_1 \cdot x = e_2, \quad e_2 \cdot x = -e_3; \qquad x \cdot x = pe_1 + qe_2 - ae_3.$$

Let $e_4 = x - ae_2 - q/2e_1$. Then

$$(4.3) \qquad \begin{array}{lll} e_2 \cdot e_3 = e_1, & e_3 \cdot e_3 = e_2; & e_4 \cdot e_1 = e_2, \\ e_1 \cdot e_4 = e_2, & e_2 \cdot e_4 = -e_3; & e_4 \cdot e_4 = pe_1, \\ & \text{and all other products are } 0. & \end{array}$$

For all $p \in \mathbf{R}$, L has a Lie algebra structure; $[e_2, e_3] = e_1$, $[e_2, e_4] = -e_3$. If $p = 0$, we denote the l.s. algebra given in (4.3) by L_0. If $p \neq 0$, then by letting $q = \sqrt[5]{p}$ and new "e_1" $= q^3 e_1$, "e_2" $= q^2 e_2$, "e_3" $= qe_3$, and "e_4" $= e_4/q$, we can assume $p = 1$ in (4.3). We denote this by L_1. It is easy to check that L_0 and L_1 are not isomorphic keeping in mind that $R = L_\infty$ is a characteristic ideal. (The example D. Fried obtained is L_0.) We conclude as follows.

4.3 Theorem. *There are two (nonisomorphic) nilpotent l.s. algebras without translations (i.e. $T = 0$). These are given by*

$$L_0 = \langle e_1, e_2, e_3, e_4 \mid e_2 \cdot e_3 = e_1,\ e_3 \cdot e_3 = e_2,\ e_4 \cdot e_1 = e_2,\ e_1 \cdot e_4 = e_2,\ e_2 \cdot e_4 = -e_3 \rangle,$$

$$L_1 = \langle e_1, e_2, e_3, e_4 \mid e_2 \cdot e_3 = e_1,\ e_3 \cdot e_3 = e_2,\ e_4 \cdot e_1 = e_2,\ e_1 \cdot e_4 = e_2,\ e_2 \cdot e_4 = -e_3,\ e_4 \cdot e_4 = e_1 \rangle.$$

5. Classification

In this section we briefly sketch the outline of how to get the final list for the classification of 4-dimensional nilpotent l.s. algebra L. There are five different types of L according to the dimension of $C(L)$.

(1) $\dim C(L) = 0$: In this case necessarily $T(L) = 0$ and we have two isomorphism classes found in Theorem 4.3.

(2) $\dim C(L) = 1$: L is an extension of the form $0 \to \mathbf{R} \to L \to K \to 0$; $[g] \in E(K, \mathbf{R})$, and K can be any of the 3-dimensional nilpotent l.s. algebras. This is the most complicated case since we have to calculate $H^2(K, \mathbf{R})$ and $\mathrm{Aut}(K)$ for all 3-dimensional K and compute the orbit space as in 3.3. We refer the reader to [14] for these calculations.

(3) $\dim C(L) = 2$: L is an extension of the form $0 \to \mathbf{R}^2 \to L \to K \to 0$; $[g] \in E(K, \mathbf{R}^2)$, where $K = \mathbf{R}^2$ or N^2. The classification of this type is easier than the previous one and again we refer the reader to [14] for the details.

(4) $\dim C(L) = 3$: L is an extension of the form $0 \to \mathbf{R}^3 \to L \to \mathbf{R} \to 0$, $[g] \in E(\mathbf{R}, \mathbf{R}^3)$. In this case it is easy to show that there is only one isomorphism class.

(5) $\dim C(L) = 4$: L is just $\mathbf{R}^4$, the trivial l.s. algebra.

As is well known, there are three 4-dimensional nilpotent Lie algebras up to isomorphism (see, for example [5] or [6]). We will denote these as follows.

$$
\begin{aligned}
&A = \langle x_1, x_2, x_3, x_4 \mid - \rangle; \quad \text{abelian},\\
(5.1)\qquad &H = \langle x_1, x_2, x_3, x_4 \mid [x_2, x_3] = x_1 \rangle; \quad \text{Heisenberg} \oplus \mathbf{R},\\
&T = \langle x_1, x_2, x_3, x_4 \mid [x_2, x_3] = x_1,\ [x_3, x_4] = x_2 \rangle.
\end{aligned}
$$

We can recognize these three types easily from the observation that A, H, and T are 1, 2 and 3-step respectively, or that the dimension of $[LL]$ is 0, 1, and 2 respectively.

5.1 Theorem. *Let L be a nilpotent l.s. algebra of dimension 4. Then, up to isomorphism, we have the following table. (In the table, structure of L is written with respect to a basis $\mathbf{e} = \{e_1, e_2, e_3, e_4\}$ presenting only nontrivial products.)*

5.2 Remark. (i) The table already shows the classification of 3-dimensional nilpotent l.s. algebras K as listed in (2)(a)–(i) at the titles.

(ii) Apply Proposition 3.2 to obtain the nontrivial products from $[h] \in E(K,C) \subset H^2(K,C)$ in the table below.

(iii) In order to convert the l.s. algebra to the Lie subgroup of Aff($\mathbf{R}^4$) which acts simply transitively on $\mathbf{R}^4$, first read off the left multiplication $\lambda : L \to \mathfrak{gl}(L)$ from the nonzero products and then exponentiate $(\lambda_x, x) \in \mathfrak{aff}(\mathbf{R}^4)$ to obtain the corresponding Lie group element. (Recall Proposition 1.1 for this process.)

(1) $\dim C = 0$:

	Nonzero product	(Lie algebra type)
1.	$e_2 \cdot e_3 = e_1,\ e_3 \cdot e_3 = e_2, \begin{cases} e_1 \cdot e_4 = e_2 \\ e_4 \cdot e_1 = e_2 \end{cases}, e_2 \cdot e_4 = -e_3.$	(T)
2.	$e_2 \cdot e_3 = e_1,\ e_3 \cdot e_3 = e_2, \begin{cases} e_1 \cdot e_4 = e_2 \\ e_4 \cdot e_1 = e_2 \end{cases}, e_2 \cdot e_4 = -e_3,\quad e_4 \cdot e_4 = e_1.$	(T)

(2) $\dim C = 1$: $0 \to \mathbf{R} \to L \to K \to 0$; $[h] \in E(K, \mathbf{R})$.

(a) $K = \mathbf{R}^3 = \langle b_1, b_2, b_3 \mid - \rangle$

$[h]$ (h: $K \times K \to \mathbf{R}$ is given as a matrix with respect to a basis $b = (b_1, b_2, b_3)$).

3.	$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$; $e_2 \cdot e_2 = e_1,\ e_3 \cdot e_3 = e_1,\ e_4 \cdot e_4 = e_1.$	(A)
4.	$\begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix}$; $e_2 \cdot e_2 = e_1,\ e_3 \cdot e_3 = e_1,\ e_4 \cdot e_4 = -e_1.$	(A)
5.	$\begin{bmatrix} 1 & 1 & \\ -1 & 0 & 1 \\ & 1 & \end{bmatrix}$; $e_2 \cdot e_2 = e_1, \begin{cases} e_2 \cdot e_3 = e_1 \\ e_3 \cdot e_2 = -e_1 \end{cases}, \begin{cases} e_3 \cdot e_4 = e_1 \\ e_4 \cdot e_3 = e_1 \end{cases}.$	(H)
6.	$\begin{bmatrix} & 1 & \\ -1 & 0 & 1 \\ & 1 & \end{bmatrix}$; $\begin{cases} e_2 \cdot e_3 = e_1 \\ e_3 \cdot e_2 = -e_1 \end{cases}, \begin{cases} e_3 \cdot e_4 = e_1 \\ e_4 \cdot e_3 = e_1 \end{cases}.$	(H)
7_t.	$\begin{bmatrix} 1 & 1 & \\ -1 & t & \\ & & 1 \end{bmatrix}$; $e_2 \cdot e_2 = e_1, \begin{cases} e_2 \cdot e_3 = e_1 \\ e_3 \cdot e_2 = -e_1 \end{cases}, e_3 \cdot e_3 = te_1,\ e_4 \cdot e_4 = e_1.$	(H)
8_t.	$\begin{bmatrix} 1 & 1 & \\ -1 & t & \\ & & -1 \end{bmatrix}$ $t \geqslant 0$; $e_2 \cdot e_2 = e_1,\ e_2 \cdot e_3 = e_1,\ e_3 \cdot e_3 = te_1,\ e_4 \cdot e_4 = -e_1.$	(H)

(b) $K = K_1$, $0 \to \mathbf{R} \to K_1 \to \mathbf{R}^2 \to 0$ with $g = [^1{}_1]$, or $K_1 = \langle b_1, b_2, b_3 \mid b_2 \cdot b_2 = b_1,\ b_3 \cdot b_3 = b_1 \rangle$.

9.	$\begin{bmatrix} 0 & 1 & 0 \\ & & \\ & & \end{bmatrix}$; $e_2 \cdot e_3 = e_1,\ e_3 \cdot e_3 = e_2,\ e_4 \cdot e_4 = e_2.$	(H)

10. $\begin{bmatrix} 0 & 1 & 0 \\ & & 0 \\ & & 1 \end{bmatrix}$; $e_2 \cdot e_3 = e_1,\ e_3 \cdot e_3 = e_2,\ e_4 \cdot e_4 = e_1 + e_2.$ (H)

11_t. $\begin{bmatrix} 0 & 1 & 0 \\ & & 1 \\ & & t \end{bmatrix}$ $t \geqslant 0;\ e_2 \cdot e_3 = e_1,\ e_3 \cdot e_4 = e_1,\ e_3 \cdot e_3 = e_2,\ e_4 \cdot e_4 = te_1 + e_2.$ (H)

(c) $K = K_2$, $0 \to \mathbf{R} \to K_2 \to \mathbf{R}^2 \to 0$ with $g = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$, or $K_2 = \langle b_1, b_2, b_3 \mid b_2 \cdot b_2 = b_1,\ b_3 \cdot b_3 = -b_1 \rangle$.

12. $\begin{bmatrix} 0 & 1 & 0 \\ & & \\ & & \end{bmatrix}$; $e_2 \cdot e_3 = e_1,\ e_3 \cdot e_3 = e_2,\ e_4 \cdot e_4 = -e_2.$ (H)

13. $\begin{bmatrix} 0 & 1 & 1 \\ & & \\ & & \end{bmatrix}$; $e_2 \cdot e_3 = e_1,\ e_2 \cdot e_4 = e_1,\ e_3 \cdot e_3 = e_2,\ e_4 \cdot e_4 = -e_2.$ (H)

14. $\begin{bmatrix} 0 & 1 & 1 \\ & & 1 \\ & & \end{bmatrix}$; $e_2 \cdot e_3 = e_1,\ e_2 \cdot e_4 = e_1,\ e_3 \cdot e_4 = e_1,\ e_3 \cdot e_3 = e_2,\ e_4 \cdot e_4 = -e_2$ (H)

15. $\begin{bmatrix} 0 & 1 & 1 \\ & & 1 \\ & & 1 \end{bmatrix}$; $e_2 \cdot e_3 = e_1,\ e_2 \cdot e_4 = e_1,\ e_3 \cdot e_4 = e_1,\ e_3 \cdot e_3 = e_2,\ e_4 \cdot e_4 = e_1 - e_2$ (H)

16. $\begin{bmatrix} 0 & 1 & 0 \\ & & 0 \\ & & 1 \end{bmatrix}$; $e_2 \cdot e_3 = e_1,\ e_3 \cdot e_3 = e_2,\ e_4 \cdot e_4 = e_1 - e_2.$ (H)

17_t. $\begin{bmatrix} 0 & 1 & 0 \\ & & 1 \\ & & t \end{bmatrix}$; $e_2 \cdot e_3 = e_1,\ e_3 \cdot e_4 = e_1,\ e_3 \cdot e_3 = e_2,\ e_4 \cdot e_4 = te_1 - e_2.$ (H)

(d) $K = K_3(\lambda)$, $0 \to \mathbf{R} \to K_3 \to \mathbf{R}^2 \to 0$ with $g = \begin{bmatrix} 1 & 1 \\ -1 & \lambda \end{bmatrix}$, or $K_3 = \langle b_1, b_2, b_3 \mid b_2 \cdot b_2 = b_1,\ b_2 \cdot b_3 = b_1,\ b_3 \cdot b_3 = \lambda b_1 \rangle$.

18_λ. $\begin{bmatrix} 0 & 1 & 1 \\ 0 & & \\ -2 & & \end{bmatrix}$; $e_2 \cdot e_3 = e_1,\ e_2 \cdot e_4 = e_1,\ e_3 \cdot e_4 = e_2,\ e_3 \cdot e_3 = e_2,\ e_4 \cdot e_2 = -2e_1,\ e_4 \cdot e_3 = -e_2,\ e_4 \cdot e_4 = \lambda e_2.$ (T)

19_λ. $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ -2 & 1 & 0 \end{bmatrix}$; $e_2 \cdot e_3 = e_1,\ e_2 \cdot e_4 = e_1,\ e_3 \cdot e_4 = e_2,\ e_3 \cdot e_3 = e_2,\ e_4 \cdot e_2 = -2e_1,\ e_4 \cdot e_3 = e_1 - e_2,\ e_4 \cdot e_4 = \lambda e_2.$ (T)

$20_{\lambda,t}$. $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ -2 & t & 0 \end{bmatrix}$; $e_2 \cdot e_3 = e_1,\ e_2 \cdot e_4 = e_1,\ e_3 \cdot e_4 = e_2,\ e_3 \cdot e_3 = e_1 + e_2,\ e_4 \cdot e_2 = -2e_1,\ e_4 \cdot e_3 = te_1 - e_2,\ e_4 \cdot e_4 = \lambda e_2.$ (T)

(e) $K = K_4$, $0 \to \mathbf{R} \to K_4 \to \mathbf{R}^2 \to 0$ with $g = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, or $K_4 = \langle b_1, b_2, b_3 \mid b_2 \cdot b_3 = b_1,\ b_3 \cdot b_2 = -b_1 \rangle$.

21. $\begin{bmatrix} 0 & 1 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$; $e_2 \cdot e_3 = e_1,\ e_3 \cdot e_4 = e_2,\ e_3 \cdot e_2 = -2e_1,\ e_4 \cdot e_3 = -e_2.$ (T)

22. $\begin{bmatrix} 0 & 1 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $e_2 \cdot e_3 = e_1,\ e_3 \cdot e_4 = e_2,\ e_4 \cdot e_4 = e_1,$ $e_3 \cdot e_2 = -2e_1,\ e_4 \cdot e_3 = -e_2.$ (T)

23. $\begin{bmatrix} 0 & 1 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & -1 \end{bmatrix}$; $e_2 \cdot e_3 = e_1,\ e_2 \cdot e_4 = e_1,\ e_3 \cdot e_4 = e_2,\ e_4 \cdot e_4 = e_1,$ $e_3 \cdot e_2 = -2e_1,\ e_4 \cdot e_2 = -2e_1,\ e_4 \cdot e_3 = -e_2.$ (T)

(f) $K = K_5$, $0 \to \mathbf{R}^2 \to K_5 \to \mathbf{R} \to 0$ with $g(1, 1) = e_1$, or $K_5 = \langle b_1, b_2, b_3 \mid b_3 \cdot b_3 = b_1 \rangle$.

24. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$; $e_2 \cdot e_3 = e_1,\ e_4 \cdot e_4 = e_2.$ (H)

25. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$; $e_2 \cdot e_3 = e_1,\ e_3 \cdot e_4 = e_1,\ e_4 \cdot e_4 = e_2.$ (H)

26. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$; $e_2 \cdot e_3 = e_1,\ e_4 \cdot e_3 = e_1,\ e_4 \cdot e_4 = e_2.$ (H)

27. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$; $e_2 \cdot e_3 = e_1,\ \begin{cases} e_3 \cdot e_4 = e_1 \\ e_4 \cdot e_3 = e_1 \end{cases},\ e_4 \cdot e_4 = e_2.$ (H)

28. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$; $e_2 \cdot e_4 = e_1,\ e_3 \cdot e_3 = e_1,\ e_4 \cdot e_4 = e_2.$ (H)

29. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$; $e_2 \cdot e_4 = e_1,\ e_4 \cdot e_3 = e_1,\ e_4 \cdot e_4 = e_2.$ (H)

30_t. $\begin{bmatrix} 0 & 0 & t \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$; $\begin{cases} e_2 \cdot e_3 = te_1 \\ e_4 \cdot e_2 = e_1 \end{cases},\ e_3 \cdot e_3 = e_1,\ e_4 \cdot e_4 = e_2.$ $\begin{cases} A \text{ if } t = 1. \\ H \text{ if } t \neq 1. \end{cases}$

31_t. $\begin{bmatrix} 0 & 0 & t \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$; $\begin{cases} e_2 \cdot e_3 = te_1 \\ e_4 \cdot e_2 = e_1 \end{cases},\ e_3 \cdot e_4 = e_1,\ e_4 \cdot e_4 = e_2.$ (H)

(g) $K = K_6$, $0 \to \mathbf{R} \to K_6 \to N \to 0$ with $g = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$, or $K_6 = \langle b_1, b_2, b_3 \mid b_2 \cdot b_3 = b_1,\ b_3 \cdot b_3 = b_2 \rangle$.

32. $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$; $e_3 \cdot e_2 = e_1,\ e_3 \cdot e_4 = e_2,\ e_4 \cdot e_4 = e_3.$ (T)

33. $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$; $e_3 \cdot e_2 = e_1,\ \begin{cases} e_3 \cdot e_4 = e_2 \\ e_4 \cdot e_3 = e_1 \end{cases},\ e_4 \cdot e_4 = e_3.$ (T)

34. $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$; $e_3 \cdot e_2 = e_1,\ \begin{cases} e_3 \cdot e_4 = e_2 \\ e_4 \cdot e_3 = -e_1 \end{cases},\ e_4 \cdot e_4 = e_3.$ (T)

35. $\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$; $\begin{cases} e_2 \cdot e_4 = -e_1 \\ e_4 \cdot e_2 = e_1 \end{cases},\ e_3 \cdot e_4 = e_2,\ e_4 \cdot e_4 = e_3.$ (T)

36. $\begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$; $e_2 \cdot e_4 = 2e_1,\ e_3 \cdot e_3 = e_1,\ e_3 \cdot e_4 = e_2,\ e_4 \cdot e_4 = e_3,$ $e_4 \cdot e_2 = -e_1,\ e_4 \cdot e_3 = e_1.$ (T)

$37_t.$ $\begin{bmatrix} 0 & 0 & 1-t \\ 0 & 1 & 0 \\ t & 0 & 0 \end{bmatrix}$; $e_2 \cdot e_4 = (1-t)e_1,\ e_3 \cdot e_3 = e_1,\ e_3 \cdot e_4 = e_2,$ $e_4 \cdot e_4 = e_3,\ e_4 \cdot e_2 = te_1,$ (H if $t = 1/2$, T if $t \neq 1/2$)

$38_t.$ $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & t & 0 \end{bmatrix}$; $e_2 \cdot e_4 = e_1,\ e_3 \cdot e_2 = e_1,\ e_3 \cdot e_4 = e_2,$ $e_3 \cdot e_3 = e_1,\ e_4 \cdot e_3 = te_1,\ e_4 \cdot e_4 = e_3.$ (T)

(h) $K = K_7(\mu)$, $0 \to \mathbf{R} \to K_7 \to N \to 0$ with $g = [\begin{smallmatrix} 0 & 1 \\ \mu & 0 \end{smallmatrix}]$, or $K_7 = \langle b_1, b_2, b_3 \mid b_2 \cdot b_3 = b_1,\ b_3 \cdot b_2 = \mu b_1,\ b_3 \cdot b_3 = b_2 \rangle$.

$39_\mu.$ $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$; $e_3 \cdot e_3 = e_1,\ e_4 \cdot e_2 = e_1,\ e_3 \cdot e_4 = e_2,$ $e_4 \cdot e_4 = e_3,\ e_4 \cdot e_3 = \mu e_2.$ (Case $\mu = 1$: H) (Case $\mu \neq 1$: T)

$40_\mu.$ $\begin{bmatrix} 0 & 0 & 1 \\ 0 & (1+\mu)/2 & 0 \\ (-+3\mu)/2 & 1 & 0 \end{bmatrix}$; $e_2 \cdot e_4 = e_1,\ e_3 \cdot e_4 = e_2,$ $e_4 \cdot e_2 = (-1+3\mu)/2e_1,\ e_4 \cdot e_3 = e_1 + \mu e_2,$ $e_3 \cdot e_3 = (1+\mu)/2e_1,\ e_4 \cdot e_4 = e_3.$
(Case $\mu = 1$: H)
(Case $\mu \neq 1$: T)

$41_{\mu,t}.$ $\begin{bmatrix} 0 & 0 & 1 \\ 0 & t & 0 \\ t-1+\mu & 0 & 0 \end{bmatrix}$, $t \neq (1+\mu)/2$; $e_2 \cdot e_4 = e_1,\ e_3 \cdot e_4 = e_2,$ $e_4 \cdot e_2 = (t-1+\mu)e_1,\ e_4 \cdot e_3 = \mu e_2,$ $e_3 \cdot e_3 = te_1,\ e_4 \cdot e_4 = e_3.$
(Case $\mu = 1$: A if $t = 1$, H if $t \neq 1$)
(Case $\mu \neq 1$: H if $t = 2 - \mu$, T if $t \neq 1$)

(i) $K = K_8$, $0 \to \mathbf{R} \to K_8 \to N \to 0$ with $g = [\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}]$, or $K_8 = \langle b_1, b_2, b_3 \mid b_3 \cdot b_2 = b_1,\ b_3 \cdot b_3 = b_2 \rangle$.

42. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$; $e_4 \cdot e_2 = e_1,\ e_4 \cdot e_3 = e_2,\ e_4 \cdot e_4 = e_3.$ (T)

43. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$; $e_4 \cdot e_2 = e_1,\ \begin{cases} e_3 \cdot e_4 = e_1 \\ e_4 \cdot e_3 = e_2 \end{cases},\ e_4 \cdot e_4 = e_3.$ (T)

$44_t.$ $\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ t & 0 & 0 \end{bmatrix}$; $\begin{cases} e_2 \cdot e_4 = e_1 \\ e_4 \cdot e_2 = te_1 \end{cases},\ e_3 \cdot e_3 = -e_1,\ e_4 \cdot e_3 = e_2,\ e_4 \cdot e_4 = e_3.$ (H if $t = 1$, T if $t \neq 1$)

$45_t.$ $\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 1 \\ t & 0 & 0 \end{bmatrix}$; $\begin{cases} e_2 \cdot e_4 = e_1 \\ e_4 \cdot e_2 = te_1 \end{cases},\ e_3 \cdot e_3 = -e_1,$ $\begin{cases} e_3 \cdot e_4 = e_1 \\ e_4 \cdot e_3 = e_2 \end{cases},\ e_4 \cdot e_4 = e_3.$ (H if $t = 1$, T if $t \neq 1$)

(3) $\dim C = 2$: $0 \to \mathbf{R}^2 \to L \to K \to 0$; $[h] \in E(K, \mathbf{R}^2)$.
(a) $K = \mathbf{R}^2 = \langle b_1, b_2 \mid - \rangle$.

46. $\begin{bmatrix} 0 & e_2 \\ -e_2 & 0 \end{bmatrix}$; $\begin{cases} e_3 \cdot e_4 = e_2 \\ e_4 \cdot e_3 = -e_2 \end{cases}.$ (H)

47. $\begin{bmatrix} e_1 & e_1 \\ -e_1 & 0 \end{bmatrix}$; $e_3 \cdot e_3 = e_1$, $\begin{cases} e_3 \cdot e_4 = e_1, \\ e_4 \cdot e_3 = -e_1. \end{cases}$ (H)

48. $\begin{bmatrix} e_1 & e_2 \\ -e_2 & 0 \end{bmatrix}$; $e_3 \cdot e_3 = e_1$, $\begin{cases} e_3 \cdot e_4 = e_2, \\ e_4 \cdot e_3 = -e_2. \end{cases}$ (H)

49. $\begin{bmatrix} e_1 & e_2 \\ -e_2 & e_1 \end{bmatrix}$; $e_3 \cdot e_3 = e_1$, $\begin{cases} e_3 \cdot e_4 = e_2 \\ e_4 \cdot e_3 = -e_2 \end{cases}$, $e_4 \cdot e_4 = e_1$. (H)

50. $\begin{bmatrix} e_1 & e_2 \\ -e_2 & -e_1 \end{bmatrix}$; $e_3 \cdot e_3 = e_1$, $\begin{cases} e_3 \cdot e_4 = e_2 \\ e_4 \cdot e_3 = -e_2 \end{cases}$, $e_4 \cdot e_4 = -e_1$. (H)

51_t. $\begin{bmatrix} e_1 & te_1 \\ -te_1 & e_1 \end{bmatrix}$ $t \geqslant 0$; $e_3 \cdot e_3 = e_1$, $\begin{cases} e_3 \cdot e_4 = te_1 \\ e_4 \cdot e_3 = -te_1 \end{cases}$, $e_4 \cdot e_4 = e_1$. (A if $t = 0$, H if $t > 0$)

52_t. $\begin{bmatrix} e_1 & te_1 \\ -te_1 & -e_1 \end{bmatrix}$ $t \geqslant 0$; $e_3 \cdot e_3 = e_1$, $\begin{cases} e_3 \cdot e_4 = te_1 \\ e_4 \cdot e_3 = -te_1 \end{cases}$, $e_4 \cdot e_4 = -e_1$. (A if $t = 0$, H if $t > 0$)

53_t. $\begin{bmatrix} e_1 & (1+t)e_2 \\ (1-t)e_2 & -e_1 \end{bmatrix}$ $t \geqslant 0$; $e_3 \cdot e_3 = e_1$, $\begin{cases} e_3 \cdot e_4 = (1+t)e_2 \\ e_4 \cdot e_3 = (1-t)e_2 \end{cases}$, $e_4 \cdot e_4 = e_1$. (A if $t = 0$, H if $t > 0$)

54. $\begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix}$; $e_3 \cdot e_3 = e_1$, $e_4 \cdot e_4 = e_2$. (A)

55_t. $\begin{bmatrix} e_1 & e_1 + te_2 \\ -e_1 - te_2 & e_2 \end{bmatrix}$; $e_3 \cdot e_3 = e_1$, $\begin{cases} e_3 \cdot e_4 = e_1 + te_2 \\ e_4 \cdot e_3 = -e_1 - te_2 \end{cases}$, $e_4 \cdot e_4 = e_2$. (H)

56. $\begin{bmatrix} e_1 & e_1 + e_2 \\ -e_1 + e_2 & 0 \end{bmatrix}$; $e_3 \cdot e_3 = e_1$, $\begin{cases} e_3 \cdot e_4 = e_1 + e_2 \\ e_4 \cdot e_3 = -e_1 + e_2 \end{cases}$, (H)

57_t. $\begin{bmatrix} e_1 & (1+t)e_2 \\ (1-t)e_2 & 0 \end{bmatrix}$; $e_3 \cdot e_3 = e_1$, $\begin{cases} e_3 \cdot e_4 = (1+t)e_2 \\ e_4 \cdot e_3 = (1-t)e_2 \end{cases}$, ($A$ if $t = 0$, H if $t > 0$)

(b) $K = N^2 = \langle b_1, b_2 \mid b_2 \cdot b_2 = b_1 \rangle$.

58. $\begin{bmatrix} 0 & 0 \\ e_1 & 0 \end{bmatrix}$; $e_4 \cdot e_3 = e_1$, $e_4 \cdot e_4 = e_3$. (H)

59. $\begin{bmatrix} 0 & e_1 \\ e_2 & 0 \end{bmatrix}$; $\begin{cases} e_3 \cdot e_4 = e_1 \\ e_4 \cdot e_3 = e_2 \end{cases}$ $e_4 \cdot e_4 = e_3$. (H)

60_t. $\begin{bmatrix} 0 & e_1 \\ te_1 & 0 \end{bmatrix}$; $\begin{cases} e_3 \cdot e_4 = e_1 \\ e_4 \cdot e_3 = te_1 \end{cases}$ $e_4 \cdot e_4 = e_3$. (A if $t = 1$, H if $t \neq 1$)

(4) $\dim C = 3$; $0 \to \mathbf{R}^3 \to L \to \mathbf{R} \to 0$, $h(1,1) = e_1$.

61. $e_4 \cdot e_4 = e_1$. (A)

(5) $\dim C = 4$; $L = \mathbf{R}^4$.

62. — (A)

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